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Meromorphic Structure of the Mellin Transforms and Short-Distance Behavior of Correlation Integrals

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The short-distance behavior of the measure of a sphere and of the correlation integral is determined, in the case of disconnected repellers, by scaling laws whose corrections are oscillating functions, periodic or aperiodic, depending on exact or approximate self-similarity of the measure. The Mellin transforms prove to be the correct analytic tool in order to investigate these corrections to scaling. It has been previously proved that they are meromorphic for linear Cantor sets and that the leading pole gives the correlation dimension in agreement with the results of the thermodynamic formalism. Here we show that the residues of these poles can also be computed to any desired accuracy with simple algorithms and that the knowledge of the singularity spectrum of the Mellin transforms provides the Fourier spectrum of the scaling correction for the self-similar measure and that it reproduces the damped oscillations in the generic case. The method applies to the nonlinear repellers such as the disconnected Julia sets by using an approximation theorem.

KEY WORDS: Mixing repellers; Mellin transform; potential and energy integrals; corrections to scaling laws; singularity spectrum and residues.

1. INTRODUCTION

Strange attractors[1,2] and repellers[3] play a crucial role in nonlinear dynamics in the new approach to turbulence based on the excitation of a few nonlinear modes rather than infinitely many linear modes.[4,5] A large number of systems whose attractors are the tensor product of Cantor sets and smooth manifolds and whose dynamics have a hyperbolic character have been investigated. From a mathematical point of view the repellers are

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simpler; they arise as basin boundaries of attracting fixed points and are attractors for the inverse map.

We shall restrict our analysis to the mixing repellers[6] on the real line, which are Cantor sets endowed with an ergodic measure invariant with respect to an expanding map. The Julia sets[7] are the most important nonlinear repellers; the linear Cantors are the simplest repellers and the building blocks to describe the Julia sets themselves. The natural setting in which to describe these systems is provided by the thermodynamical formalism.[8–10] In the case of linear Cantor sets explict results are obtained for the Gibbs measures[11,12]; the convergence to the thermodynamic functions of Julia sets of sequences computed from linear Cantors has also been proved.[13,14]

The local behavior of the measure and the short-distance behavior of the correlation integrals of the measure are not described by the thermodynamic limit, except for the leading scaling behavior. The corrections to the scaling laws were first numerically investigated[15] and qualitatively explained with a scaling argument for self-similar repellers.[16] The first analytical treatment was based on the energy integral given by Mellin transform of the correlation[17,18] following a procedure which was successful in analyzing the measures of the Julia sets.[19] The energy integrals were shown to be meromorphic for the linear Cantor sets and generalized to describe the dimension spectrum.[20] From the possible patterns of poles the possible behaviors of the corrections to the scaling law of the correlation functions were outlined.[21]

In the present work an extension of the method to investigate the behavior of the measure of a ball of radius l as $l \to 0$ is proposed, by introducing the potential function (a detailed analysis will be found in ref. 21) and an algorithm to compute the residues of the poles for both the potential and the energy integrals.

The residues are obtained from the functional equation satisfied by the potential and energy integrals; they are related to integrals on the invariant measure which are approximated in two distinct ways by using the order-n partitions of the Cantor sets. The convergence as $n \to \infty$ is proved and in one case an explicit error bound is provided and numerically verified. No analytical estimates of the decay law of the residues were obtained, and a numerical investigation was carried out by comparing them with the $\log l$ Fourier spectrum of the correction to the scaling laws for the measure of a ball and the correlation integral for self-similar measures. The good agreement further proves that the proposed method is accurate and reliable, and the evidence of a power-law decrease of the Fourier coefficients with exponent between 1.2 and 1.4 shows that the corrections to the scaling laws are not smooth functions, at least in the periodic case.

The behavior of the correlation integrals of non-self-similar measures for some linear Cantor and Julia sets was also examined; in this case the residue spectrum is the only available tool to analyze the aperiodic corrections to the scaling law.

Numerical difficulties can be found in the search for poles when the scales are quite different and perturbation expansions with respect to the equal scale solutions do not converge. Similarly, in the case of Julia sets the method is fairly accurate when the approximation with linear Cantor sets is rapidly convergent, as for the quadratic map $F(z) = z^2 - p$ with $p \gg 2$.

The plan of the work is the following: in Section 2 we introduce the basic notations and definitions and recall some results based on scaling arguments. In Section 3 we determine the poles of the potential at the fixed points of a linear Cantor set, compute the $f(\alpha)$ spectrum, and provide the basic formulas for the residues with error estimates and convergence proofs, and their extensions to the energy integrals. In Section 4 the Fourier spectrum of the correction to the scaling law is computed for the measure of the ball and the correlation integral in the periodic case and its decay law is investigated. The generic case for the correlation integral is illustrated by some examples.

2. POTENTIAL AND ENERGY INTEGRALS FOR MIXING REPELLERS

We consider a dynamical system $(\mathscr{E}, T, \mu)$, where $\mathscr{E} \subset [0, 1]$ is the maximal invariant set of a map T defined on $[0, 1]$, and μ is an invariant ergodic measure. If $T^{-1}([0, 1])$ is the union of disconnected sets and T is expanding on $T^{-1}([0, 1])$, we call $\mathscr{E}$ a *repeller*.[6] Letting

$$T^{-1}([0, 1]) = \bigcup_{j=1}^{s} A_j, \qquad A_j \cap A_k = \varnothing \quad \text{for } j \neq k \tag{2.1}$$

the inverse of T on any of the sets A_j is defined and is denoted by T_j^{-1}, and the following notation will be used:

$$T^{-1}(A) = \bigcup_{j=1}^{s} T_j^{-1}(A), \qquad A \subset [0, 1] \tag{2.2}$$

The set $\mathscr{E}$ is a Cantor set; it is a linear Cantor set if the map T is linear on $T^{-1}([0, 1])$. A measure μ defined on $\mathscr{E}$ is invariant if $\mu(A) = \mu(T^{-1}A)$, $\forall A \subset \mathscr{E}$, and balanced with weights

$$\mathbf{p} = (p_1, \ldots, p_s), \qquad p_j \geqslant 0, \qquad p_1 + \cdots + p_s = 1 \tag{2.3}$$

if for any positive measurable set A

$$\mu(T_j^{-1}(A)) = p_j \mu(A) \tag{2.4}$$

Equation (2.4) holds if and only if for any measurable function

$$\int_{\mathscr{E}} f(x)\, d\mu(x) = \sum_{j=1}^{s} p_j \int_{\mathscr{E}} f(T_j^{-1}x)\, d\mu(x) \tag{2.5}$$

We shall denote by $(\mathscr{C}, L, \mu)$ a linear Cantor system and by $L_j^{-1}(x)$ the inverses of the map L,

$$L_j^{-1}(x) = \lambda_j x + b_j, \qquad j = 1,..., s \tag{2.6}$$

and call $0 < \lambda_j < 1$ the scales of the map. We introduce the intervals

$$\mathscr{I}_n = \bigcup_{k_1,..., k_n} I_{k_1,..., k_n}, \qquad I_{k_1,..., k_n} = L_{k_n}^{-1} \cdots L_{k_1}^{-1}([0, 1]) \tag{2.7}$$

and the partitions

$$\mathscr{A}_n = \mathscr{I}_n \cap \mathscr{C} = \bigcup_{k_1,..., k_n} A_{k_1,..., k_n}, \qquad A_{k_1,..., k_n} = I_{k_1,..., k_n} \cap \mathscr{C} \tag{2.8}$$

Letting $\mu^{(n)}$ be a sequence of Stjeltjes measures whose density is constant on each $I_{k_1,..., k_n}$, zero elsewhere, the invariant measure turns out to be defined by requiring that $\mu^{(n)}(A_{k_1,..., k_n}) = p_{k_1} \cdots p_{k_n}$. The invariant measure on $\mathscr{C}$ is then defined by

$$\mu(A_{k_1,..., k_n}) = \mu^{(n)}(A_{k_1,..., k_n}) = p_{k_1} \cdots p_{k_n} \tag{2.9}$$

Denoting by $B(x, l) \equiv (x - l, x + l)$ the sphere of center x and radius l, we introduce its Mellin transform

$$V(x; \alpha) = \int_0^\infty l^{-\alpha}\, d\mu(B(x, l)) = \int_{\mathscr{E}} |x - y|^{-\alpha}\, d\mu(y) \tag{2.10}$$

The divergence abscissa of $V(x; \alpha)$ determines the scaling exponent of $\mu(B(x, l))$, while the corrections to the scaling law are determined by the remaining singularities.

We consider as well the correlation integrals defined by

$$C(l) = \int_{\mathscr{E}} \int_{\mathscr{E}} \vartheta(l - |x - y|)\, d\mu(x)\, d\mu(y) = \int_{\mathscr{E}} \mu(B(x, l))\, d\mu(x) \tag{2.11}$$

and their Mellin transforms

$$\Phi(\alpha)=\int_0^\infty l^{-\alpha}\,dC(l)=\int_{\mathscr{E}} V(x;\alpha)\,d\mu(x)$$

$$=\int_{\mathscr{E}}\int_{\mathscr{E}} |x-y|^{-\alpha}\,d\mu(x)\,d\mu(y) \tag{2.12}$$

We recall that, according to Young's theorem,[22] if the limit of $\log \mu(B(x,l))/\log l$ exists on a set $\hat{\mathscr{E}}\subset\mathscr{E}$ of full μ measure, then the limit is constant in $\hat{\mathscr{E}}$ and is equal to the information dimension $D_1(\mu)$. The correlation integral scales with the correlation dimension $D_2(\mu)$, which is the limit of $\log C(l)/\log l$ as $l\to 0$.

Scaling Arguments. We first remark that for a self-similar Cantor, where $\lambda_j=\lambda$, $p_j=p\equiv 1/s$ for $1\leqslant j\leqslant s$, the corrections to the scaling laws are obtained with simple arguments. Indeed, suppose that a function scales according to $g(\lambda l)=pg(l)$; then, letting $G(x)=\log g(e^x)$, one can prove that $g(l)=l^D f(\log l)$, where $f(x)$ is periodic with period $\log\lambda$ and $D=\log p/\log\lambda$. Following the argument quoted in ref. 23 for a Cantor with equal scales, we find, using $\mu(L[A])=s\mu(A)$,

$$\mu(B(x,\lambda l))=\frac{1}{s}\,\mu\,(L[B(x,\lambda l)])=\frac{1}{s}\,\mu(B(L(x),l)) \tag{2.13}$$

As a consequence, if x is a fixed point of order n, namely $x=L^{\circ n}(x)$, from (2.13) iterated n times we obtain $\mu(B(x,\lambda^n l))=(1/s^n)\,\mu(B(x,l))$ and therefore $\mu(B(x,l))\,l^{-D}$, where $D=-\log s/\log\lambda$, is a periodic function of $\log l$ with period $n\log\lambda$.

3. FUNCTIONAL EQUATION, POLES, AND RESIDUES OF THE POTENTIAL

Using the balance properties (2.4) and (2.5) of the measure n times, we obtain the following equation for the potential:

$$V(L_{j_n}^{-1}\cdots L_{j_1}^{-1}(x);\alpha)$$

$$=\sum_{k_1,\ldots,k_n} p_{k_1}\cdots p_{k_n}\int_{\mathscr{E}} |L_{j_n}^{-1}\cdots L_{j_1}^{-1}(x)-L_{k_n}^{-1}\cdots L_{k_1}^{-1}(y)|^{-\alpha}\,d\mu(y)$$

$$=p_{j_1}\cdots p_{j_n}\lambda_{j_1}^{-\alpha}\cdots\lambda_{j_n}^{-\alpha}V(x;\alpha)+E_{j_1\cdots j_n}(x;\alpha) \tag{3.1}$$

where $E_{j_1 \cdots j_n}(x;\alpha)$ is an entire function of α (the distance between preimages of any point is strictly positive) given by

$$E_{j_1 \cdots j_n}(x;\alpha) = \sum_{k_1,\ldots,k_n \neq j_1,\ldots,j_n} p_{k_1} \cdots p_{k_n} \times \int_{\mathscr{E}} |L_{j_n}^{-1} \cdots L_{j_1}^{-1}(x) - L_{k_n}^{-1} \cdots L_{k_1}^{-1}(y)|^{-\alpha} \, d\mu(y) \quad (3.2)$$

Letting $x_{j_1,\ldots,j_n}$ be the fixed point of the map $L_{j_n}^{-1} \cdots L_{j_1}^{-1}(x)$, we have

$$V(x_{j_1,\ldots,j_n};\alpha) = \frac{E_{j_1 \cdots j_n}(x_{j_1,\ldots,j_n};\alpha)}{1 - p_{j_1} \cdots p_{j_n}(\lambda_{j_1} \cdots \lambda_{j_n})^{-\alpha}} \quad (3.3)$$

As a consequence, the singularities of $V(x_{j_1,\ldots,j_n};\alpha)$ are poles located at

$$\alpha = \frac{\sum_{l=1}^{n} \log p_{j_l}}{\sum_{l=1}^{n} \log \lambda_{j_l}} + i \frac{2\pi m}{\sum_{l=1}^{n} \log \lambda_{j_l}} \quad (3.4)$$

Denoting by $n_1,\ldots, n_s$, where $n_1 + \cdots + n_s = n$, the number of times $\lambda_1,\ldots, \lambda_s$ appear in the sum (3.4) and by $\mathbf{w} = (w_1,\ldots, w_s)$ the corresponding fractions, we see that the real part $\alpha(\mathbf{w})$ of (3.4) is given by

$$\alpha(\mathbf{w}) = \frac{\sum_{j=1}^{s} w_j \log p_j}{\sum_{j=1}^{s} w_j \log \lambda_j} \quad (3.5)$$

In the limit $n \to \infty$ the measure of the set $\mathscr{E}(\mathbf{w})$, with $\mathscr{E}(\mathbf{w})$ given by the union of the intervals $I_{j_1,\ldots,j_s}$ where the frequency of the indices $1,\ldots, s$ is $\mathbf{w}$, is zero except for $\mathbf{w} = \mathbf{p}$, and $\alpha(\mathbf{p})$ is the information dimension or Hausdorff dimension of the measure. Moreover, the Hausdorff dimension of any other set $\mathscr{E}(\mathbf{w})$ where $\alpha(\mathbf{w})$ has a constant value (the measure of these sets is zero) is given by

$$\beta = \frac{\sum_{j=1}^{s} w_j \log w_j}{\sum_{j=1}^{s} w_j \log \lambda_j} \quad (3.6)$$

and it is not difficult to show that $\beta = f(\alpha) = q\alpha - \tau(q)$, where $f(\alpha)$ is defined according to ref. 24; the result explicitly quoted in ref. 24 for $s = 2$ agrees with our results.

The present analysis show that $\mu(B(x, l))\, l^{-\alpha}$, where α here denotes the real part of (3.4) for any fixed point x, is a periodic function of $\log l$ with period given by 2π divided by the imaginary part of (3.4). As a consequence, we extend the results given by the scaling arguments for equal scales to the generic case. Moreover the computation of the residues can be explicitly carried out. In the Appendix we compute analytically the residues

for the map defined by $L_1^{-1}(x) = x/2$, $L_2^{-1}(x) = (x+1)/2$, with $p_1 = p_2 = 1/2$, for which μ is the Lebesgue measure on $[0, 1]$. In this case $\alpha = 1 + i2\pi m/\log 2$ and we check that all the residues vanish except for the real pole; moreover, the computation shows that the residues do not depend on the interval $[0, R]$ where the Mellin transform of $\mu(B(x, l))$ is performed provided that R is strictly positive and less than 1. The independence of the residues from the cutoff R is a general result, as shown in the Appendix, and proves that they are related only to the local properties of the measure.

Denoting by $\alpha = \underline{\alpha} + im\eta$ the poles (3.4) of the potential at a fixed point of order n, we have that the Mellin inverse transform is given by

$$\mu(B(x, l)) = l^{\underline{\alpha}} \sum_{m=-\infty}^{+\infty} \frac{E(x; \underline{\alpha} + im\eta)}{\sum_{k=1}^{n} \log \lambda_{jk}} \exp(im\eta \log l) \tag{3.7}$$

where we used the short-hand notation x, E, $\underline{\alpha}$ for $x_{j_1,\ldots,j_n}$, $E_{j_1,\ldots,j_n}$, $\underline{\alpha}_{j_1,\ldots,j_n}$. The residues of the poles are indeed given by $E(x; \underline{\alpha} + im\eta)/\sum_{k=1}^{n} \log \lambda_{jk}$ and to compute them we must approximate E. We consider two basic approximations of the integral of a function with respect to the measures which are based on

$$\int_{\mathscr{E}} f(x)\, d\mu(x) = \sum_{k_1,\ldots,k_m} \int_{A_{k_1,\ldots,k_m}} f(x)\, d\mu(x) \tag{3.8}$$

where we apply either a mean value theorem or we replace $\mu(x)$ with $\mu^{(m)}(x)$ given by (2.9). In the former case, assuming f to be Lipschitz in $[0, 1]$ with Lipschitz constant M in $[0, 1]$ and letting x^* be any point in $I_{k_1,\ldots,k_m}$, we have

$$\begin{gathered} \int_{A_{k_1,\ldots,k_m}} f(x)\, d\mu(x) = f(x^*)\, p_{k_1} \cdots p_{k_m} + \delta_{k_1,\ldots,k_m} \\ |\delta_{k_1,\ldots,k_m}| \leqslant M p_{k_1} \cdots p_{k_m} \lambda_{k_1} \cdots \lambda_{k_m} \end{gathered} \tag{3.9}$$

We shall choose x^* to be the fixed point $x_{k_1,\ldots,k_m}$ of $L_{k_m}^{-1} \cdots L_{k_1}^{-1}(x)$ in the interval $I_{k_1,\ldots,k_m}$ to obtain

$$\begin{gathered} \int_{\mathscr{E}} f(x)\, d\mu(x) = \sum_{k_1,\ldots,k_m} p_{k_1} \cdots p_{k_m} f(x_{k_1,\ldots,k_m}) + \delta_m \\ |\delta_m| \leqslant M(p_1 \lambda_1 + \cdots + p_s \lambda_s)^m \end{gathered} \tag{3.10}$$

We shall refer to (3.10) as the ergodic quadrature formula, since we replace the integral on the measure with a weighted sum on the orbit. The

convergence rate is geometric and better than $\lambda_{\max}^m$. The second method consists in replacing μ with $\mu^{(m)}$, where

$$d\mu^{(m)}(x) = (p_{k_1} \cdots p_{k_m})/(\lambda_{k_1} \cdots \lambda_{k_m})\, dx, \qquad x \in A_{k_1, \ldots, k_m} \tag{3.11}$$

and extending $d\mu^{(m)}(x)$ defined by the rhs to $I_{k_1, \ldots, k_m}$. As a consequence, we have

$$\int_{A_{k_1, \ldots, k_m}} f(x)\, d\mu(x) = \frac{p_{k_1} \cdots p_{k_m}}{\lambda_{k_1} \cdots \lambda_{k_m}} \int_{I_{k_1, \ldots, k_m}} f(x)\, dx + \delta'_{k_1, \ldots, k_m} \tag{3.12}$$

In this case we have no explicit bound for the error $\delta'_{k_1, \ldots, k_m}$.

This method is useful to evaluate the remainder of the functional equation for the potential, since $f(x) = (\lambda x + c)^{-\alpha}$, where λ and c are such that $\lambda x + c$ never vanishes on $[0, 1]$. As a consequence,

$$\int_{\mathscr{E}} (\lambda x + c)^{-\alpha}\, d\mu(x) = \frac{1}{1-\alpha} \sum_{k_1, \ldots, k_m} \frac{p_{k_1} \cdots p_{k_m}}{\lambda_{k_1} \cdots \lambda_{k_m}} \times [(\lambda b_{k_1, \ldots, k_m} + c)^{1-\alpha} - (\lambda a_{k_1, \ldots, k_m} + c)^{1-\alpha}] + \delta'_m \tag{3.13}$$

where $I_{k_1, \ldots, k_m} \equiv [a_{k_1, \ldots, k_m}, b_{k_1, \ldots, k_m}]$ and the convergence as $m \to \infty$ is ensured.

Other approximation methods could be used if the probabilistic algorithm[25] to generate the Cantor is used, as proved in ref. 26.

For the energy integral the procedure is similar. Using twice the balance property of the measure, we have

$$\begin{aligned} \Phi(\alpha) &= \sum_{j,k=1}^{s} p_k p_j \int_{\mathscr{E}} \int_{\mathscr{E}} |L_k^{-1} x - L_j^{-1} y|^{-\alpha}\, d\mu(x)\, d\mu(y) \\ &= \sum_{k=1}^{s} p_k^2 \lambda_k^{-\alpha} \Phi(\alpha) + E(\alpha) \end{aligned} \tag{3.14}$$

where

$$E(\alpha) = \sum_{j \neq k} p_k p_j \int_{\mathscr{E}} \int_{\mathscr{E}} |L_k^{-1} x - L_j^{-1} y|^{-\alpha}\, d\mu(x)\, d\mu(y) \tag{3.15}$$

We notice that for $k \neq j$ the distance of $L_k^{-1} x$ from $L_j^{-1} y$ is always finite and therefore $E(\alpha)$ is an entire function of α. The singularities of $\Phi(\alpha)$ are poles given by the equation

$$\sum_{k=1}^{s} p_k^2 \lambda_k^{-\alpha} = 1 \tag{3.16}$$

The Frostman theorem[27] asserts that the divergence abscissa, given by the pole with lowest real part, is a lower bound to the Hausdorff dimension which is reached for the equilibrium measure. If the correlation integral scales as $C(l) \sim l^{D_2(\mu)}$, then the divergence abscissa is the correlation dimension $D_2(\mu)$.

The residues in this case are obtained by applying twice the above approximation schemes (3.8)–(3.13) to the double integral in the rhs of (3.15), and the convergence and the error bounds are similarly obtained.

4. FOURIER SPECTRA AND DECAY LAW

Numerical Results. The results on the singularity spectrum have been compared with the Fourier spectra obtained from the numerical

Table I. The Fourier Spectrum of the Function $S(y)$ Computed Numerically from the FFT and from the Residues of the Potential Function Computed through the Ergodic Method and through the Lebesgue Approximation $\mu^{(n)}$ to the Measure Are Compared for the Ternary Cantor

m	a_m	b_m
Amplitudes FFT with 1024 points		
0	0.99776	0.00000
1	0.10446	−0.04273
2	−0.10637	−0.04814
3	0.01395	−0.05512
4	−0.00848	−0.00865
5	−0.00337	−0.01990
Amplitudes ergodic theorem		
0	0.99893	0.00000
1	0.10425	−0.04371
2	−0.10755	−0.04534
3	0.01178	−0.05572
4	−0.00897	−0.00834
5	−0.00456	−0.01981
Amplitudes approximated measure		
0	0.99892	0.00000
1	0.10424	−0.04371
2	−0.10756	−0.04535
3	0.01177	−0.05574
4	−0.00897	−0.00835
5	−0.00456	−0.01983

computation of the measure of a ball for a generic Cantor system and of the correlation integral for a self-similar Cantor.

Indeed, the correction to the scaling law of the correlation integral is periodic when the scales are equal, since the poles are equally spaced on a line parallel to the imaginary axis. If the scales are different but satisfy a resonance condition, namely the vector $\mathbf{v} = (\log \lambda_1, ..., \log \lambda_s)$ is rationally dependent, then there are still poles equally spaced on a line parallel to the imaginary axis, but the correlation is periodic only asymptotically, since there can be further poles to the right of this line. In the generic case the poles can be spread in the half plane to the right of the convergence abscissa and $C(l)$ exhibits damped oscillations.

In the latter cases the only possible comparison is between the numerical correlation and the inverse transform of the energy integral approximated with the leading singularities.

We consider the Cantor with equal scales and weights: $\lambda_1 = \lambda_2 = 1/3$, $p_1 = p_2 = 1/2$, and the Cantor with two different scales $\lambda_1 = 1/3$, $\lambda_2 = 1/5$ and equal weights. We first discuss the results obtained for the measure of a ball, since in this case the correction to the scaling law is always a periodic function. Letting $\bar{x}$ be a fixed point of the map, we have computed the function $S(\log l) = \mu(B(\bar{x}, l))\, l^{-\sigma}$, where σ is the scaling exponent given by (3.5). The function $S(y)$ evaluated numerically turns out to be periodic and its Fourier spectrum was obtained using a FFT algorithm. The first five coefficients are quoted in Table I and compared with the coefficients obtained from the residues of the poles $\alpha = \sigma + in\eta$ of the potential function. These residues were computed using either the ergodic theorem or the

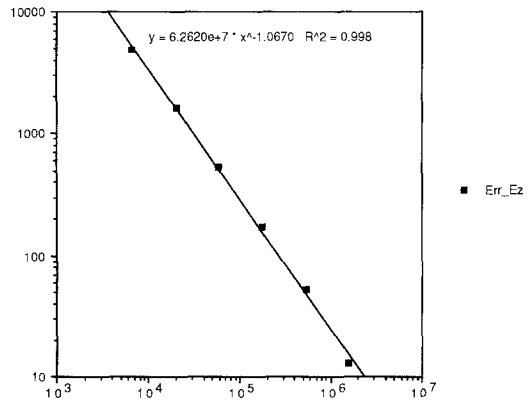

Fig. 1. Plot of the logarithm of error on the residues of the potential function at the real pole for the ternary Cantor against $\log \lambda^n$, where n is the order of the partition used in the approximation.

approximation of the measure $\mu(x)$ with a Lebesgue measure $\mu^{(n)}(x)$ and are also quoted in Table I. It can be observed that the Fourier coefficients obtained from the residues with two different approximation methods agree within five decimal places, while the agreement with the numerical Fourier spectrum is within three to four decimal places.

This is consistent with the error estimates: indeed, the error in the FFT with $M = 2^m$ is $\propto M^{-1}$ and in our case $M = 1024$ was used. The error estimate with the ergodic approximation was λ^n if the order-n partitions were used. In Fig. 1 a plot of the logarithm of the error against $\log \lambda^n$ is shown and the data sitting on a straight line confirm the estimate.

In Fig. 2 we show the function $S(y)$ computed with 30 Fourier components: the function obtained from the original data and from the Fourier spectrum computed from the residues are indistinguishable.

The decay law of the spectrum was also analyzed. In Fig. 3a we show the coefficients between $30 < m < 512$ computed from the FFT: the decay obtained from the first 512 coefficients is fitted with a power law $k^{-1.25}$ and the exponent varied by less than 7% when the spectrum obtained from the residues is considered; see Fig. 3b. The fit obtained by cutting the first coefficients up to the 30th gives an exponent whose variation is within 0.05.

For the second example of a Cantor with two different scales the situation is very similar as far as the spectrum obtained from the FFT and the residues is concerned. In Table II the first components are quoted and it can be noticed that the agreement is slightly worse than in the self-similar case; the decay law of the spectrum is similar. The case of the Julia set is not consider here, for the following reason: we know that we can

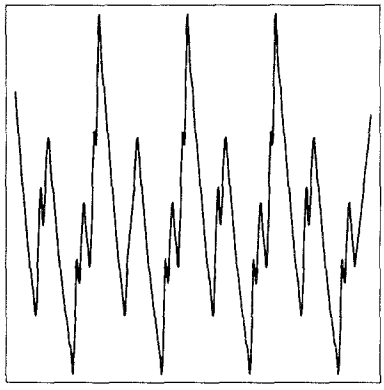

Fig. 2. Plot of the function $S(y)$ computed numerically for the Cantor with scales $\lambda_1 = 1/3$, $\lambda_2 = 1/5$. The plot obtained from the first 30 Fourier components of its Fourier spectrum is indistinguible. Axes: $-25 \leqslant x = \textit{abscissa} \leqslant 0$; $0.5 \leqslant y = \textit{ordinate} \leqslant 1.2$.

approximate the measure of the Julia set with the measure of a linear Cantor $L^{(n)}$ whose first partition agrees with the order-n partition of the Julia. However, since the measure is a very local property, the only meaningful comparison concerns averages of the measure on finite intervals. The comparison of the scaling exponent and the correction of the scaling laws at any finite order will be significantly different. The problem does not exist for the correlation integral, which is an average over the measures of the balls. In this case the sequence of linear Cantor sets provides a good approximation to the correlation integral of the Julia set, as shown in ref. 21.

The correlation integral has been analyzed following the same scheme. Let $\bar{S}(y)$ be the correction to the scaling law, namely $\bar{S}(\log l) = C(l)\, l^{-D_2}$.

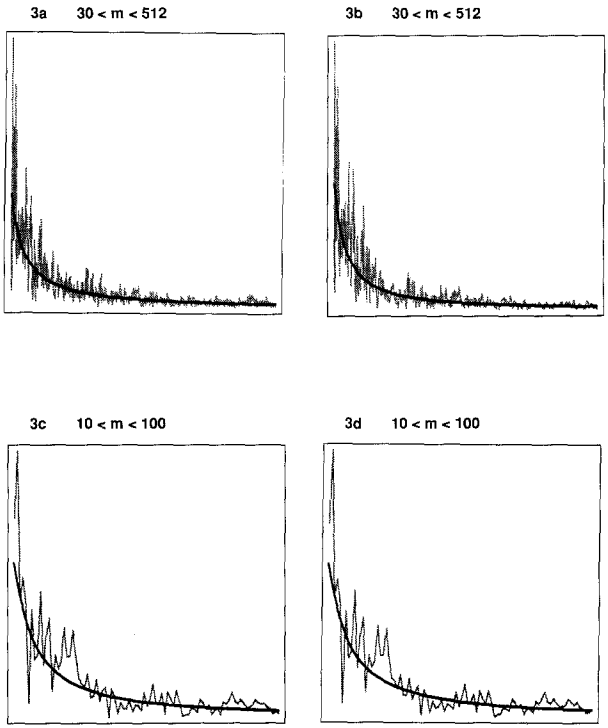

Fig. 3. (a) Plot of the Fourier spectrum of $S(y)$ computed from the FFT for the ternary Cantor set. Axes: $30 \leqslant x \leqslant 512$; $0 \leqslant y \leqslant 0.005$. (b) Plot of the Fourier spectrum of $S(y)$ computed from the residues calculated with the method of the approximation of the measure for the ternary Cantor set. Axes: $30 \leqslant x \leqslant 512$; $0 \leqslant y \leqslant 0.005$. (c) Plot of the Fourier spectrum of $\bar{S}(y)$ computed from the FFT for the ternary Cantor set. Axes: $10 \leqslant x \leqslant 100$; $0 \leqslant y \leqslant 0.0025$. (d) Plot of the Fourier spectrum of $\bar{S}(y)$ computed from the residues calculated with the ergodic quadrature formula for the ternary Cantor set. Axes: $10 \leqslant x \leqslant 100$; $0 \leqslant y \leqslant 0.0025$.

In the case of the self-similar Cantor the first Fourier coefficients computed from the FFT and from the residues are compared in Table III. Here the accuracy is slightly lower, within three decimal digits, due to the double integration required to compute the residues. However, the function $\bar{S}(y)$ obtained with 30 Fourier components turns out to be indistinguishable if the FFT or the residues results are used. In Figs. 3c and 3d the spectrum decay computed from FFT and the residues is shown, and a fit with a power law gives $k^{-1.4}$ with a difference in the exponent by less than 5%, depending on the data used.

Finally, the unequal-scales case has been examined. Here the only possible comparison is between the numerical $\bar{S}(y)$ and the inverse Mellin transform of the energy integral approximated with the leading singularities.

In order to find such singularities, we have considered Cantor sets with two scales different, but close enough to allow the use of a perturbative method.

Table II. The Same as Table I for a Cantor with Scales $\lambda_1 = 1/3$ and $\lambda_2 = 1/5$ and Equal Weights

m	a_m	b_m
Amplitudes FFT with 1024 points		
0	0.80563	0.00000
1	0.11715	0.01738
2	0.05791	−0.12699
3	0.06102	0.01178
4	0.00862	−0.02105
5	−0.01293	−0.03485
Amplitudes ergodic theorem		
0	0.81324	0.00000
1	0.11852	0.01462
2	0.04953	−0.13143
3	0.06281	0.00578
4	0.00611	−0.2190
5	−0.03492	−0.02966
Amplitudes approximated measure		
0	0.81318	0.00000
1	0.11855	0.01454
2	0.04963	−0.13142
3	0.06275	0.00583
4	0.00607	−0.02190
5	−0.03489	−0.02967

Table III. The Fourier Spectrum of the Function $\bar{S}(y)$ Computed Numerically from the FFT and from the Residues of the Correlation Function Computed through the Ergodic Method Are Compared for the Ternary Cantor

m	a_m	b_m
Amplitudes FFT with 1024 points		
0	0.96779	0.00000
1	0.03179	−0.04059
2	−0.01530	−0.00286
3	0.01462	0.00159
4	−0.00013	−0.00357
5	−0.00463	−0.00005
Amplitudes ergodic theorem		
0	0.96543	0.00000
1	0.03547	−0.03746
2	−0.01442	−0.00586
3	0.01354	0.00558
4	0.00129	−0.00339
5	−0.00404	−0.00223

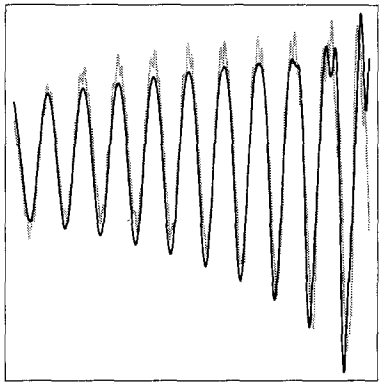

Fig. 4. Plot of the function $C(l)/l^{\nu}$ computed numerically and as the inverse Mellin transform of the energy integral approximated with the real pole $\alpha = D_2$ and the nearest two complex poles for the Cantor with the two scales $\lambda_1 = 0.\bar{3}$ and $\lambda_2 = 0.38\bar{3}$. Axes: $-16 \leqslant x \leqslant 0$; $0.96 \leqslant y \leqslant 1.06$.

Denoting by $\lambda_1 \leqslant \lambda_2$ the two scales and considering $p_1 = p_2 = 1/2$, Eq. (3.16), which gives the singularities, takes the form

$$\lambda_1^{-\alpha} + \lambda_2^{-\alpha} = 4 \tag{4.1}$$

We considered $\varepsilon = (\lambda_2 - \lambda_1)/\lambda_1$, so that $\lambda_2 = (1+\varepsilon)\lambda_1$, and we expanded the poles in a series of ε, $\alpha = \sum_{j=0}^{\infty} \varepsilon^j \alpha_j$, where α_0 is the pole in the case $\lambda_1 = \lambda_2$.

Equation (4.1) can be written as

$$1 + (1+\varepsilon)^{-\alpha} = 4\lambda_1^{\alpha} \tag{4.2}$$

The coefficients α_j were determined up to an order $j=4$ by solving Eq. (4.2). In computing the residues with the ergodic quadrature formula, we have used the first two couples of complex conjugate poles in (3.15). The oscillations so obtained, superimposed with the ones provided by the numerical computation of the correlation integral of the set, are shown in Fig. 4.

In Fig. 5 we show the first three couples of poles in the complex plane for different values of λ_2 with $\lambda_1 = 1/3$. The poles on the line parallel to the imaginary axis are the ones obtained for $\lambda_2 = \lambda_1$; the other poles are obtained for $\lambda_2 = 0.38\bar{3}$.

The Julia set for the quadratic map $F(z) = z^2 - p$ has also been considered for large values of p, such as $p = 10$. In this case the system is well approximated by a linear Cantor set with four maps and two different scales having the first-order partition identical with $F^{\circ 2}(z)$; the correlation

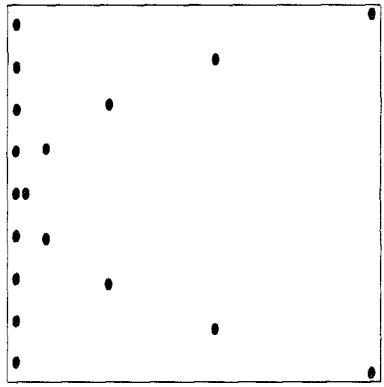

Fig. 5. Complex poles of the energy integral for a Cantor with scales $\lambda_1 = 0.\bar{3}$ and λ_2 varying. The poles on the line parallel to the imaginary axis are the ones obtained for $\lambda_2 = \lambda_1$; the other poles are obtained for $\lambda_2 = 0.38\bar{3}$. Axes: $0.5 \leqslant x \leqslant 2.25$; $-25 \leqslant y \leqslant +25$.

of the Cantor is almost identical to the one of the Julia set and is well reproduced by the first leading singularities of the energy integral. Approaching $p=2$, the number of maps of the approximating linear Cantor sets increases and correspondingly so does the computational complexity.

5. CONCLUSIONS

The Mellin transforms of the measure of a ball and of the correlation integrals allow us to determine the singularity spectra, and the possibility of determining the residues of these functions, which are meromorphic for the linear Cantor sets, makes this analysis accurate and rigorous. For the case of non-self-similar measures this is the only tool to analyze the singularity spectrum, since the Fourier analysis does not apply. The extensions to higher-dimensional linear Cantor sets are straightforward and also the application to the Julia sets is possible, so that we can say that the proposed method is applicable to the disconnected repellers.

APPENDIX

In computing the Mellin transform, the upper limit of integration remains arbitrary. We can choose it equal to $R \leqslant 1$ if the support of the measure is $[0, 1]$. However, the residues turn out to be independent of R; indeed, consider, for instance, the potential defined by

$$V_R(x_i; \alpha) = \int_0^R l^{-\alpha}\, d\mu(B(x_i, l)) \tag{A1}$$

If the measure of the ball is given by

$$\mu(B(x_i, l)) = \sum_m c_m l^{\alpha_m} \tag{A2}$$

and assuming that the convergence is uniform, we have

$$V_R(x_i; \alpha) = \sum_m \frac{c_m \alpha_m}{\alpha_m - \alpha} R^{\alpha_m - \alpha} \tag{A3}$$

The residues r_m are independent of R and are given by $r_m = -c_m \alpha_m$. As a consequence, we can always take $R=1$, so that we can write

$$V(x_i; \alpha) \equiv V_{R=1}(x_i; \alpha) = \sum_m \frac{r_m}{\alpha - \alpha_m} \tag{A4}$$

and obtain $B(x_i, l)$ as the Mellin transform of $V(x_i; \alpha)$, which is determined simply by the knowledge of the poles and residues of the potential. Using well-known properties of the measure and $\mu(B(x, l)) = \mu(x+l) - \mu(x-l)$, we have

$$V_R(x; \alpha) = \int_0^R l^{-\alpha}\, d\mu(B(x, l)) = \int_{x-R}^{x+R} |x-y|^{-\alpha}\, d\mu(y) \tag{A5}$$

by using the balance properties of the measure and chosing x_i to be the fixed point of $L_i^{-1}(x)$, it is not difficult to show that if R is small enough so that $[x_i - R, x_i + R] \cap L_k^{-1}([0, 1]) = \varnothing$ for $k \neq i$,

$$\begin{aligned} V_R(x_i; \alpha) &= \int_0^1 \chi_{[x_i - R, x_i + R]}(y)\, |y - x_i|^{-\alpha}\, d\mu(y) \\ &= \int_0^1 \chi_{[x_i - R, x_i + R]}(L_i^{-1}(y))\, |L_i^{-1}(y) - L_i^{-1}(x_i)|^{-\alpha}\, d\mu(y) \\ &= p_i \lambda_i^{-\alpha} \int_{x_i - R/\lambda_i}^{x_i + R/\lambda_i} |y - x_i|^{-\alpha} + E_R(x_i; \alpha) \\ &= p_i \lambda_i^{-\alpha} V_R(x_i; \alpha) + E_R(x_i; \alpha) \end{aligned} \tag{A6}$$

where

$$\begin{aligned} E_R(x_i; \alpha) = p_i \lambda^{-\alpha} &\int_{x_i + R}^{x_i + R/\lambda_i} |y - x_i|^{-\alpha}\, d\mu(y) \\ &+ p_i \lambda^{-\alpha} \int_{x_i - R/\lambda_i}^{x_i - R} |y - x_i|^{-\alpha}\, d\mu(y) \end{aligned} \tag{A7}$$

The changes in E_R when $[x_i - R, x_i + R] \cap L_k^{-1}([0, 1]) \neq \varnothing$ are obvious. Also, the extension to the fixed points of any order is straightforward. In this case, denoting the poles

$$\alpha = \alpha_m \equiv \frac{\log p_i}{\log \lambda_i} + i \frac{2\pi m}{\log \lambda_i} \tag{A8}$$

we have the residues given by

$$r_m = -\frac{E_R(x_i; \alpha_m)}{p_i \log \lambda_i} \tag{A9}$$

We consider now an explicit example which corresponds to the Lebesgue measure in which the computation can be explicitly carried out. Assuming $\lambda_i = p_i = 1/s$ for $1 \leqslant i \leqslant s$, we have $d\mu(x) = d(x)$,

$$V_R(x; \alpha) = 2 \frac{R^{1-\alpha}}{1-\alpha} \tag{A10}$$

The computation of $E_R(x_i; \alpha)$ gives

$$E_R(x_i; \alpha) = 2R^{1-\alpha} \left(\frac{1}{s}\right)^{1-\alpha} \frac{s^{1-\alpha} - 1}{1-\alpha} \tag{A11}$$

and due to the uniformity of the measure, the result is independent of x_i. Moreover, one checks the identity $(1 - s^{\alpha - 1}) V_R(x; \alpha) = E_R(x; \alpha)$. Here we would have a sequence of poles at $\alpha_m = 1 + i2\pi m/\log s$, but it is immediate to check that $r_0 = -2$, while all the remaining residues vanish identically, as it should be.

The situation for the correlation energy and the correlation integral is completely analogous. We have the independence of the residues from R and consequently once we know poles and residues, we reconstruct $C(l)$, taking the Mellin transform with l ranging in $[0, 1]$.

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